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1977 J. Phys. A: Math. Gen. 10 L49

(http://iopscience.iop.org/0305-4470/10/3/003)

J. Phys. A: Math. Gen., Vol. 10, No. 3, 1977. Printed in Great Britain. © 1977

LETTER TO THE EDITOR

On the unitary gauge

A Kupiainen and E Byckling

Department of Technical Physics, Helsinki University of Technology, Otaniemi, Finland

Received 7 January 1977

Abstract. We give a rigorous treatment of the unitary gauge in the SU(2) Yang–Mills–Higgs system. It is shown that the massive vector fields give a long-range contribution along the negative z axis corresponding to the string in the electromagnetic field.

We consider the SU(2) Yang–Mills–Higgs system with the monopole solution given by 't Hooft (1974) and Polyakov (1974):

$$A_a^i = \epsilon_{aij} r^j A(r), \qquad \phi_a = \phi(r) r^a / r \tag{1}$$

where $A(r)$ and $\phi(r)$ tend exponentially to their vacuum values—$-1/er^2$ and a constant, respectively—as $r \to \infty$. The asymptotic Higgs field defines a mapping from a sphere in ordinary space to a sphere in isospace belonging to the homotopy class with winding number one. The so called unitary gauge is defined as the gauge in which the Higgs field has a constant direction in isospace, for instance the three-axis. This gauge has been widely used (Arafune *et al* 1975, Jackiw and Rebbi 1976, Hasenfratz and Ross 1976), because in it the massless and massive gauge fields are separated; $\boldsymbol{A}_3$ being the long-ranged photon field and $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ being the massive vector fields. However, in the unitary gauge the Higgs field defines a mapping between spheres belonging to the trivial homotopy class and thus it is clear that this gauge can only be reached by a singular gauge transformation, when starting with the solution (1). Such a transformation is for instance

$$g(\boldsymbol{r}) = \begin{pmatrix} \cos\theta/2 & e^{-i\phi}\sin\theta/2 \\ -e^{i\phi}\sin\theta/2 & \cos\theta/2 \end{pmatrix} \tag{2}$$

in spinor representation, whence

$$A_a^i \to b_a^i (er)^{-1}(1 + er^2 A(r)) \qquad a = 1, 2, \qquad A_3^i \to \frac{\epsilon_{ij3} r_j}{er(r+z)} \tag{3}$$

where b_a^i are two orthogonal vectors in isospace.

Such a gauge transformation, however, is not a proper one, because it changes the physics; (3) gives diverging energy and monopole strength zero instead of one. This happens because the magnetic field acquires the Dirac string, which will not be compensated by other terms in the energy.

In order to avoid these difficulties we define the unitary gauge as a limit (in the sense of distributions) of non-singular gauge transformations g_ϵ. g_ϵ is defined by (2) except in the cone $\pi-\epsilon \leqslant \theta \leqslant \pi$, where we demand it to be non-singular. The requirement of differentiability of g_ϵ at $\theta=\pi-\epsilon$ and unitarity gives us

$$g_\epsilon = \begin{pmatrix} f_1(\theta) & e^{-i\phi} f_2(\theta) \\ -e^{i\phi} f_2(\theta) & f_1(\theta) \end{pmatrix} \tag{4}$$

where

$$\begin{aligned} f_2(\theta) &= A + B(\epsilon-\delta) + C(\epsilon-\delta)^2, & \delta &= \pi-\theta, A = \sin\tfrac{1}{2}(\pi-\epsilon), B = \tfrac{1}{2}\cos\tfrac{1}{2}(\pi-\epsilon) \\ f_1(\theta) &= (1-f_2^2)^{1/2}, & C &= B + 2A/\epsilon. \end{aligned} \tag{5}$$

The form of g_ϵ as $\epsilon > 0$ is of course not unique, but the limit as $\epsilon \to 0$ is unique.

The gauge fields are now given by (3), except in the cone, where we get by means of (4) and (5) (neglecting terms of order $O(\epsilon)$ in $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$)

$$\begin{aligned} \boldsymbol{A}_1^\epsilon &= \frac{2}{er}\left(\cos\phi\sin\phi\left(\frac{f_1 f_2}{\sin\theta} - a(\theta)\right), \sin^2\phi\, a(\theta) - \frac{f_1 f_2 \cos^2\phi}{\sin\theta}, 0\right) \\ \boldsymbol{A}_2^\epsilon &= \frac{2}{er}\left(a(\theta)\cos^2\phi + \frac{\sin^2\phi f_1 f_2}{\sin\theta}, \left(a(\theta) - \frac{f_1 f_2}{\sin\theta}\right)\sin\phi\cos\phi, 0\right) \\ \boldsymbol{A}_3^\epsilon &= \frac{2f_2^2}{er^2\sin^2\theta}(y, -x, 0) \end{aligned} \tag{6}$$

where $a = f_2' f_1 - f_1' f_2$. By construction, gauge invariant quantities such as energy and monopole strength, will not change in the limit $\epsilon \to 0$. To see where the cancellations arise, let us calculate the magnetic field. The contribution from the $-z$ axis is given by (6)

$$\nabla \times \boldsymbol{A}_3 = -\frac{\boldsymbol{r}}{er^3} + \lim_{\epsilon\to 0} \frac{8r_i}{er^3\sin\theta}\left(\frac{\epsilon-\delta}{\epsilon^2} - \frac{(\epsilon-\delta)^3}{\epsilon^4}\right) = -\frac{4\pi}{e}\delta(x)\delta(y)\theta(-z)\boldsymbol{e}_z - \frac{\boldsymbol{r}}{er^3} \tag{7}$$

which is the Dirac monopole field. However, for the magnetic field we also have to take the term $e\boldsymbol{A}_1 \times \boldsymbol{A}_2$. For the solution (3) this vanishes exponentially, thus giving zero flux at infinity. On the other hand, from (6) we get a contribution on the negative z axis:

$$\lim_{\epsilon\to 0} e\boldsymbol{A}_1^\epsilon \times \boldsymbol{A}_2^\epsilon = \lim_{\epsilon\to 0}\frac{8\boldsymbol{e}_z}{er^2\delta}\left(\frac{(\epsilon-\delta)^3}{\epsilon^4} - \frac{\epsilon-\delta}{\epsilon^2}\right) = \frac{4\pi}{e}\delta(x)\delta(y)\theta(-z)\boldsymbol{e}_z \tag{8}$$

which exactly cancels the string in (7). This also renders the term $\frac{1}{4}F_{ij}^3 F_{ij}^3 = (\nabla \times \boldsymbol{A}_3 + e\boldsymbol{A}_1 \times \boldsymbol{A}_2)^2$ in the energy density finite. From (6) we see that the fields $\boldsymbol{A}_1^\epsilon$ and $\boldsymbol{A}_2^\epsilon$ are also finite on the negative z axis as $\epsilon \to 0$, but (8) shows that their product (as distributions) is infinite! Massive vector fields have acquired a long-range behaviour that exactly cancels the singularities. Thus gauge invariant and gauge covariant expressions can be calculated in the unitary gauge directly using the fields (3) and neglecting singularities. Other expressions have to be treated separately on the z axis by means of (6). As an example we get

$$\lim_{\epsilon\to 0}\nabla \times \boldsymbol{A}_1^\epsilon = \lim_{\epsilon\to 0} e\boldsymbol{A}_3^\epsilon \times \boldsymbol{A}_2^\epsilon \simeq \cos\phi\,\delta(x)\delta(y)\theta(-z)\boldsymbol{e}_z = 0$$

and similarly for $\boldsymbol{A}_2$.

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